B.Sc. V Sem Examination-2013
(Physics)
MODEL ANSWER - AS-2781
Basic Quantum Mechanics

Section-A

1. (i) (a) (ii) (C) (iii) (b) (iv) (a) (V) (C)
(vi) (C) (Vii) (b)

(Viii) If mass of electron is m and it is revolving in a orbit of radius r with velocity v around the nucleus, then its angular momentum is

$$P_\phi = mvr = \frac{nh}{2\pi} \quad \text{where } n = 1, 2, 3 \ldots \text{(an integer)}$$

This is called Bohr's quantum condition.

(ix) Given $\Psi = a\, e^{-i(Et - P_x x)/\hbar}$

Differentiating with respect to x

$$\frac{\partial \Psi}{\partial x} = a\, e^{-i/\hbar\,(Et - P_x x)} \times \frac{i}{\hbar} P_x$$

$$= \frac{i P_x}{\hbar} \Psi$$

or $P_x \Psi = -i\hbar \frac{\partial}{\partial x}(\Psi)$

$\therefore$ Operator for $\hat{P}_x = -i\hbar \frac{\partial}{\partial x}$.

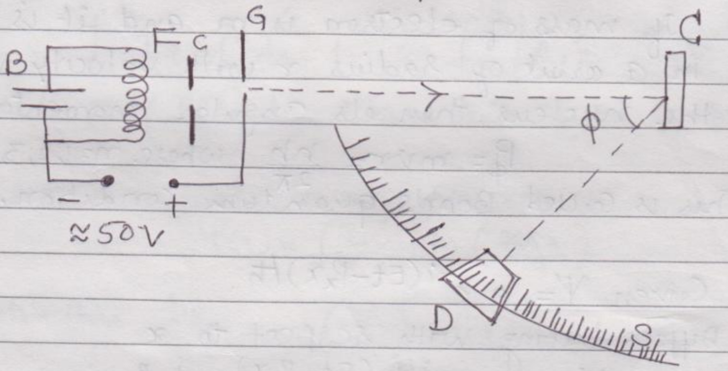
(X) The energy of simple harmonic oscillator is $E_n = (n + \frac{1}{2}) h\nu$. Hence the energy in the ground state $n=0$ is $E_0 = \frac{1}{2} h\nu$. But according to Planck's quantum theory the energy of SHO is $E_n = nh\nu$ hence ground state energy is $E_0 = 0$. Thus according to Schrodinger theory the energy of SHO in ground state ($n=0$) is $\frac{1}{2} h\nu$ and all the energy levels are shifted by $\frac{1}{2} h\nu$. This energy is called zero point energy.

②

Section-B

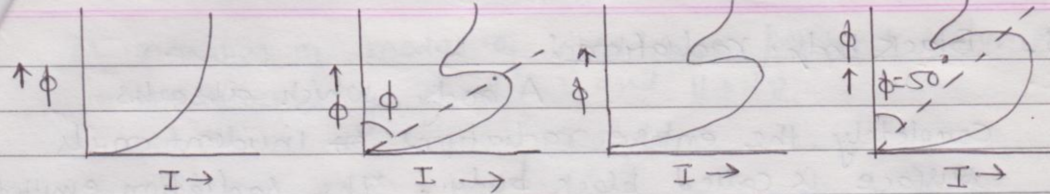
2. In 1924, Louis de-Broglie suggested that the dual nature is not only of light but each moving material particle has the dual nature. He assumed a wave to be associated with each moving material particle which is called the matter wave. The wave-length is given by $\lambda = \frac{h}{p}$.

Davisson and Germer's Experiment:-

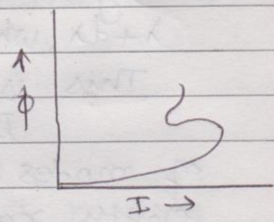


In 1925, Davisson and Germer show experimentally the existance of wave matter. In the above figure filament F is heated by a battery B to emit out electrons from it. These electrons are passed through the collimating aperture G to get a fine e^- beam. This is incident on the nickle crystal C normally on face (111). This can be rotated and the e^- beam is diffracted by this crystal which can be detected by a detector D. This detector D can be rotated on a circular scale S.

The intensity I of the diffracted electron beam is measured for different values of voltage V. The graphs are plotted for various values of accelerating voltages 40V, 44V, 48V, 54V and 60V respectively which are shown below:



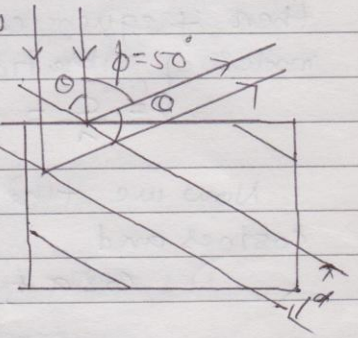
40 volt 44 volt 48 volt 54 volt



60 volt

From these graphs it is clear that at V=40 volt a smooth curve is obtained. On increasing the voltage to 44V an ~~electron~~ elevation is obtained in the curve which increases with the increase in accelerating voltage and becomes maximum at $\phi = 50^\circ$ when V= 54 volt. This concludes that there is a strong diffraction of electron beam of energy 54ev also.

From x-ray analysis we know that a nickel crystal behaves like a diffraction grating. The interplanner separation for the (1 1 1) plane is $d_{111} = 0.91\,\mathring{A}$. From fig. if the glancing angle on the crystal for the angle of diffraction $\phi = 50^\circ$ is θ, then



$$2\theta = 180^\circ - \phi$$

$$\Rightarrow \theta = \frac{180^\circ - \phi}{2} = \frac{(180^\circ - 50^\circ)}{2} = 65^\circ$$

Hence from Bragg's law $2d\sin\theta = n\lambda$, for $n=1$

$$\lambda = 2d\sin\theta = 2\times 0.91 \times \sin 65^\circ = 1.65\,\mathring{A}$$

According to de-Broglie formula $\lambda = \sqrt{\frac{150}{V}}\,\mathring{A} = 1.67\,\mathring{A}$

Since the wavelength is very close to the wavelength obtained from Bragg's relation hence it establishes the wave nature of electron beam.

3. Black body radiation :-

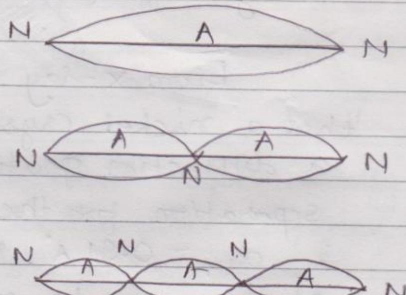
A body which absorbs completely the entire radiations ~~on~~ incident on its surface is called black body. The radiation emitted by a black body consists wavelength range λ to $\lambda + d\lambda$ which shows a continuous energy spectrum. This is called black body radiation.

In the Rayleigh-Jean theory number of modes of vibration per unit volume in the wavelength range λ and $\lambda + d\lambda$ mean energy per mode of vibration is

$$u_\lambda d\lambda = N_\lambda d\lambda \times \overline{E}$$

(i) Calculation of $N_\lambda d\lambda$:

If the speed of wave is c, then frequencies of allowed modes of vibrations will be



$$\nu = \frac{c}{\lambda} = \frac{c}{2l/n} = \frac{nc}{2l}$$

Now we find the direction cosines and

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

$$\therefore \quad n_1^2 + n_2^2 + n_3^2 = \left(\frac{2a\nu}{c}\right)^2$$

Hence the number of allowed modes of vibrations in frequency from 0 to ν per unit volume

$$= \frac{1}{8} \text{ volume of sphere of radius } \frac{2a\nu}{c}$$

$$= \frac{4}{3}\frac{\pi\nu^3}{c^3}$$

or number of modes of vibrations per unit volume in the frequency range ν and $\nu + d\nu$

$$= \frac{4\pi \nu^2 d\nu}{c^3}$$

Now since e-m waves are transverse in nature.

$$\therefore \quad N_\nu \, d\nu = 2 \times \frac{4\pi \nu^2 d\nu}{c^3}$$

$$\boxed{N_\nu \, d\nu = \frac{8\pi \nu^2 d\nu}{c^3}}$$

(ii) To find the average energy per mode of vibration —

$$\overline{E} = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E \, e^{-E/KT} \, dx \, dp}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-E/KT} \, dx \, dp}$$

After substituting $E = \frac{1}{2}Kx^2 + \frac{p^2}{2m}$ and solving the integration we get

$$E = \frac{1}{2}KT + \frac{1}{2}KT = KT$$

$\therefore$ The energy density per unit volume is

$$\boxed{u_\nu \, d\nu = \frac{8\pi \nu^2 KT \, d\nu}{c^3}}$$

4. A particle executing simple harmonic motion under a linear restoring force is called a simple harmonic oscillator.

According to quantum mechanics, the time independent one dimensional Schrödinger's equation for SHO is

(6)

$$\frac{\partial^2\psi}{\partial x^2} + \frac{2m}{\hbar^2}(E-V)\psi = 0$$

or $$\frac{\partial^2\psi}{\partial x^2} - \alpha^2 x^2\psi = -\epsilon\psi \quad \text{—(1)}$$ where $\frac{mk}{\hbar^2} = \alpha^2$ & $\frac{2mE}{\hbar^2} = \epsilon$

The asymptotic solution (when $x \to \infty$, $\alpha^2 x^2 >> \epsilon$) provide us $\frac{\partial^2\psi}{\partial x^2} - \alpha^2 x^2\psi = 0$ whose solution is $\psi = \pm e^{\pm\alpha x^2/2}$ —(2)

Among these two solutions $\psi = e^{-\alpha x^2/2}$ is possible which obeys the condition that ψ or $|\psi|^2$ decreases with increasing x. Hence the general solution of eq^n (1) is $\psi(x) = f(x)\, e^{-\alpha x^2/2}$

After calculating $\frac{\partial^2\psi}{\partial x^2}$ and substituting in eq^n (1) we get

$$\frac{\partial^2 f}{\partial x^2} - 2\alpha x\frac{\partial f}{\partial x} + (\epsilon - \alpha)f = 0$$

Let $y = \sqrt{\alpha}\,x$ and $f(x) = H(y)$ then this eq^n change in to Hermite polynomial

$$\frac{\partial^2 H}{\partial y^2} - 2y\frac{\partial H}{\partial y} + \left(\frac{\epsilon}{\alpha} - 1\right)H = 0$$

Let the solution of the above eq^n is $H_y = \sum_{p=0}^{\infty} a_p y^p$

Calculating $\frac{\partial H}{\partial y}$ and $\frac{\partial^2 H}{\partial y^2}$ we get

$$\sum p(p-1)a_p y^{p-2} - \sum\left[2p - \left(\frac{\epsilon}{\alpha} - 1\right)\right]a_p y^p = 0 \quad \text{—(3)}$$

The above equation is valid only if the coefficient of each power of y is zero ie

$$a_{p+2}(p+2)(p+1) = a_p\left(2p - \frac{\epsilon}{\alpha} + 1\right)$$

$$a_{p+2} = \frac{(2p - \epsilon/\alpha + 1)}{(p+2)(p+1)}\,a_p$$

Let the series be finite for $p = n$ then

$$2n - \frac{\varepsilon}{\alpha} + 1 = 0 \quad \text{or} \quad n = \frac{1}{2}\left(\frac{\varepsilon}{\alpha} - 1\right)$$

$$\text{or} \quad \frac{\varepsilon}{\alpha} = 2n + 1$$

after substituting values of ε and α we get finally

$$\boxed{E_n = \left(n + \frac{1}{2}\right) h\nu}$$

5. Given $\psi_{(x)} = N e^{-x^2/2a^2} e^{ikx}$

(i) Value of N :-

It's complex conjugate is $\psi^*_{(x)} = N e^{-x^2/2a^2} e^{-ikx}$

According to the condition of normalization

$$\int_{-\infty}^{\infty} \psi_{(x)} \psi^*(x)\, dx = 1$$

$$\therefore \quad N^2 \int_{-\infty}^{\infty} e^{-x^2/a^2}\, dx = 1$$

but $\int_{-\infty}^{\infty} e^{-x^2/a^2}\, dx = \sqrt{\frac{\pi}{1/a^2}} = a\sqrt{\pi}$

$$N^2 \times a\sqrt{\pi} = 1 \quad \text{or} \quad N = \frac{1}{a^{1/2}\, \pi^{1/4}}$$

(ii) Probability density :-

$$P_{(x)} = \psi_{(x)} \psi^*_{(x)} = N e^{-x^2/2a^2} e^{ikx} \times N e^{-x^2/2a^2} e^{-ikx}$$

$$= N^2 e^{-x^2/a^2} = \frac{1}{a\sqrt{\pi}}\, e^{-x^2/a^2}$$

Obviously at $x = 0$ probability is maximum. As x increases $P_{(x)}$ decreases exponentially and at $x = \pm a$ the probability decreases to $\frac{1}{e}$ times its maximum value. Hence probability is maximum in a region upto $x = a$ on either side of $x = 0$.

6. (a) If we assume that e^- resides inside the nucleus, then the maximum uncertainty in the position of electron to be inside the nucleus is $(\Delta x)_{max} \approx 2r = 2\times10^{-14}$ m.

From Heisenberg's uncertainty principle $\Delta x . \Delta p \approx \hbar$

∴ Minimum uncertainty in the momentum of electron

$$(\Delta p)_{min} = \frac{\hbar}{\Delta x} \approx \frac{1.055\times10^{-34}}{2\times10^{-14}} \approx 5.275\times10^{-21}\ kg\, m/s$$

Then from the theory of relativity if momentum of electron is p

$$E = \sqrt{p^2c^2+m_0^2c^4} \approx pc.$$

$$\text{or } E = 5.275\times10^{-21} \times 3\times10^{8} \approx 15.825\times10^{-13}\ J$$

$$\approx \frac{15.825\times10^{-13}}{1.6\times10^{-19}}\ eV \approx 9.89\times10^{6}\ eV \approx 9.89\ MeV$$

Thus if we assume the electron resides inside the nucleus its energy must be nearly 9.89 MeV whereas the maximum energy of β particles (or electrons) emitted from the nucleus is nearly 2 to 3 MeV. Thus we conclude that electrons cannot reside inside the nuclei.

(b)

Given $m = 40$ gm $= 40\times10^{-3}$ kg

and $v = 10^3$ m/s

therefore de-Broglie wavelength associated with this bullet is

$$\lambda = \frac{h}{mv} = \frac{6.6\times10^{-34}\ Js}{(40\times10^{-3}\times10^{3})}$$

$$\lambda = 1.65\times10^{-35}\ m$$

7. If $\psi(x)$ is the eigen function of the operator

$$\hat{A} = \left(\frac{\partial^2}{\partial x^2} - x^2 \right) \text{ then}$$

$$\hat{A}\psi_{(x)} = \left[\frac{\partial^2}{\partial x^2} - x^2 \right] \left(A x e^{-x^2/2} \right) = \frac{\partial^2}{\partial x^2}\left(A x e^{-x^2/2} \right) - A x^3 e^{-x^2/2}$$

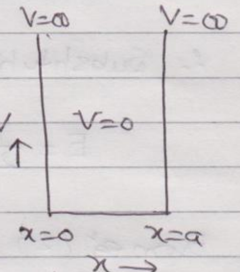
$$= A x^3 e^{-x^2/2} - 3 A x e^{-x^2/2} - A x^3 e^{-x^2/2}$$

$$= -3 A x e^{-x^2/2}$$

$$= -3\psi_{(x)}$$

This equation is analogous to $\hat{A}\psi_{(x)} = \lambda\psi_{(x)}$. Hence the given function $\psi_{(x)}$ is the eigen function of the operator $\hat{A}$ with Eigen value $\lambda = -3$.

8. Consider a single particle of mass m, velocity v is moving in side a one dimensional box along x-axis. Inside the box there is no force therefore potential energy $V_{(x)} = 0$. The walls are perfectly rigid hence the value of V abruptly increases at the walls and the time independent Schrödinger equation is



$$\frac{\partial^2\psi}{\partial x^2} + \frac{2m}{\hbar^2} E\psi = 0 \quad (1)$$

Let the solution of above equation is

$$\psi(x) = A \sin(Bx + C)$$

where A, B and C are constants which can be determined using boundary conditions.

Boundary conditions are:

$$|\psi_{(x)}|^2 = 0 \,, \text{ when } x=0 \text{ and } x=a$$

$$\text{or } |\psi_{(x)}| = 0 \quad \text{When } x=0 \text{ and } x=a$$

On applying the boundary conditions we get $\sin C = 0$ and $\sin(Ba+C) = 0$ i.e. $C=0$ and $\sin Ba = 0$ or $Ba = n\pi$

$$\therefore \psi_{(x)} = A \sin\left(\frac{n\pi x}{a}\right)$$

Using normalization condition

$$\int_0^a |\psi(x)|^2 dx = 1 \quad \text{or} \quad \int_0^a \left|A \sin\left(\frac{n\pi x}{a}\right)\right|^2 dx = 1$$

$$A^2 \int_0^a \sin^2\left(\frac{n\pi x}{a}\right) dx = 1 \text{ or } A^2 \times \frac{a}{2} = 1 \text{ or } A = \sqrt{\frac{2}{a}}$$

Hence the normalized wavefunction is

$$\psi(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)$$

with this $\frac{\partial^2 \psi(x)}{\partial x^2} = -\left(\frac{n\pi}{a}\right)^2 \psi(x)$

$\therefore$ Substituting this value in eq^n ① we get

$$E = \frac{1}{2m}\left(\frac{n\pi\hbar}{a}\right)^2 = \frac{n^2 h^2}{8ma^2} \quad -②$$

From eq^n (2) it is clear that

For $n=0$, $E=0$

For $n=1$, $E = h^2/8ma^2$

For $n=2$, $E = 4h^2/8ma^2$

For $n=3$, $E = 9h^2/8ma^2$

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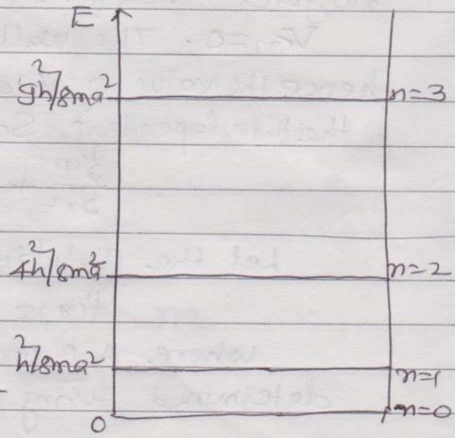

Thus the energy levels of the particle enclosed inside the box are discrete and not equi-spaced as energy follow the n^2 law.